Atmospheric Turbulence

Lecture 2, ASTR 289

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Observing through Earth's Atmosphere

- "If the Theory of making Telescopes could at length be fully brought into Practice, yet there would be certain Bounds beyond which telescopes could not perform …
- For the Air through which we look upon the Stars, is in perpetual Tremor ...
- The only Remedy is a most serene and quiet Air, such as may perhaps be found on the tops of the highest Mountains above the grosser Clouds."

Isaac Newton

Newton was right!

Outline of lecture

- Physics of turbulence in the Earth's atmosphere
  - Location
  - Origin
  - Energy sources
- Mathematical description of turbulence

Atmospheric Turbulence: Main Points

- The dominant locations for index of refraction fluctuations that affect astronomers are the atmospheric boundary layer and the tropopause (we will define these)
- Atmospheric turbulence (mostly) obeys “Kolmogorov statistics”
- Kolmogorov turbulence is derived from dimensional analysis (heat flux in = heat flux in turbulence)
- Structure functions (we will define these!) derived from Kolmogorov turbulence are $\propto r^{2/3}$
- All else will follow from these points!

Atmospheric Turbulence: Topics

- What determines the index of refraction in air?
- Origins of turbulence in Earth's atmosphere
- Energy sources for turbulence
- Kolmogorov turbulence models

Index of refraction of air

- Refractivity of air

$$N \equiv (n-1)\times 10^{6} = 77.6\left(1+\frac{7.52\,10^{-3}}{\lambda^{2}}\right)\times\left(\frac{P}{T}\right)$$

where P = pressure in millibars, T = temp. in K, λ in microns
n = index of refraction.

- Key points:
  - Index of refraction is very close to unity
  - Wavelength dependence of index of refraction is very weak!
    - » Less than 1% effect at wavelengths ~ 1 micron

Fluctuations in index of refraction are due to temperature fluctuations

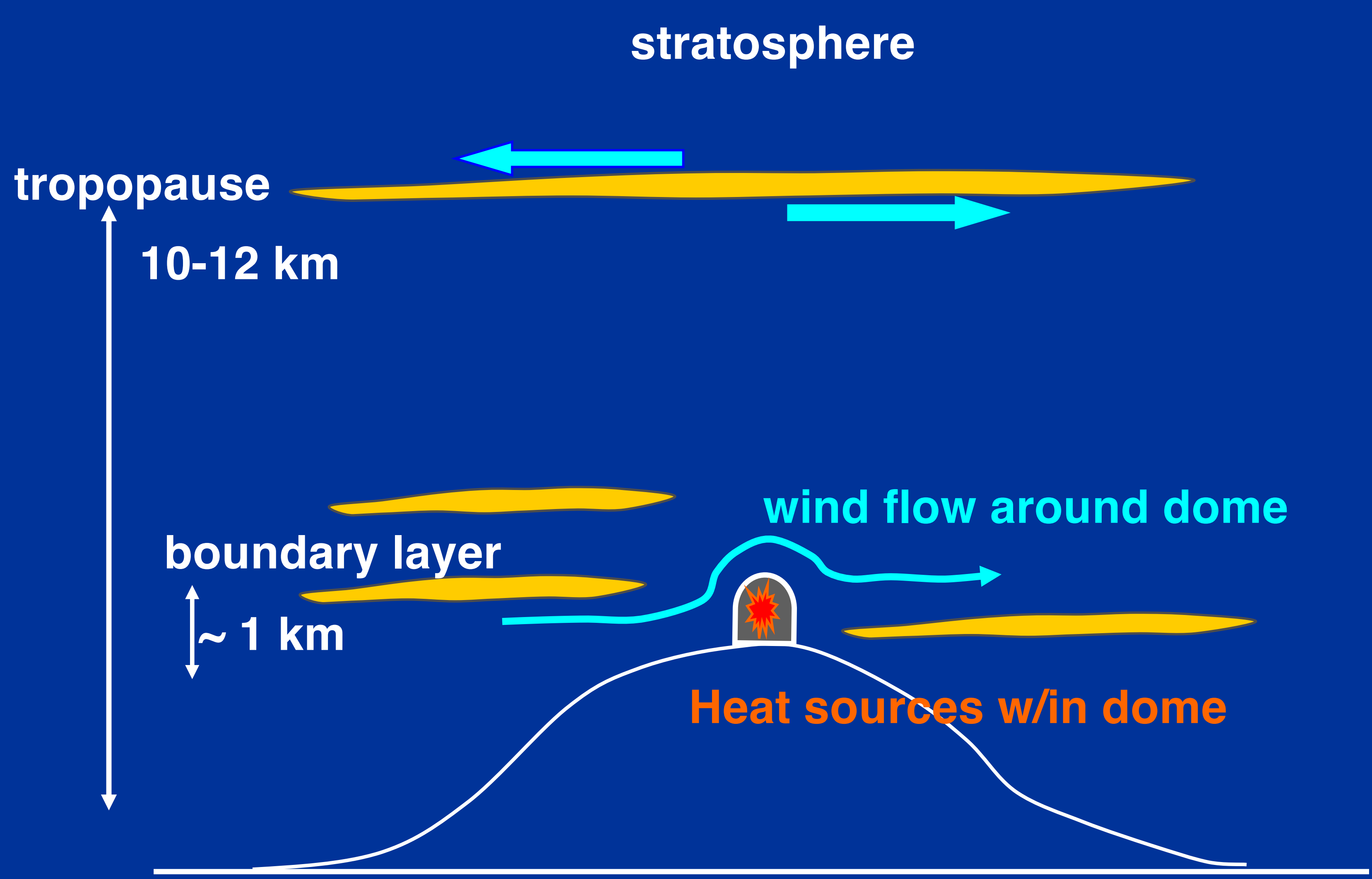
- Temperature fluctuations → index fluctuations

- Take derivative of expression on previous slide

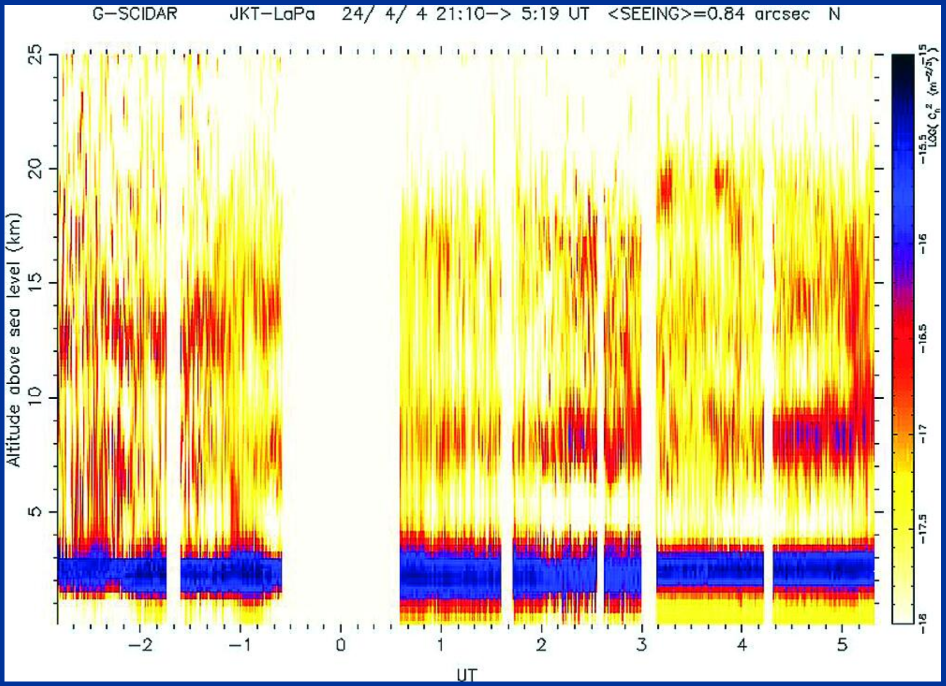
$$\delta N \cong -77.6 \times (P / T^2)\delta T$$

(pressure is constant, because velocities are highly sub-sonic -- pressure differences are rapidly smoothed out by sound wave propagation)

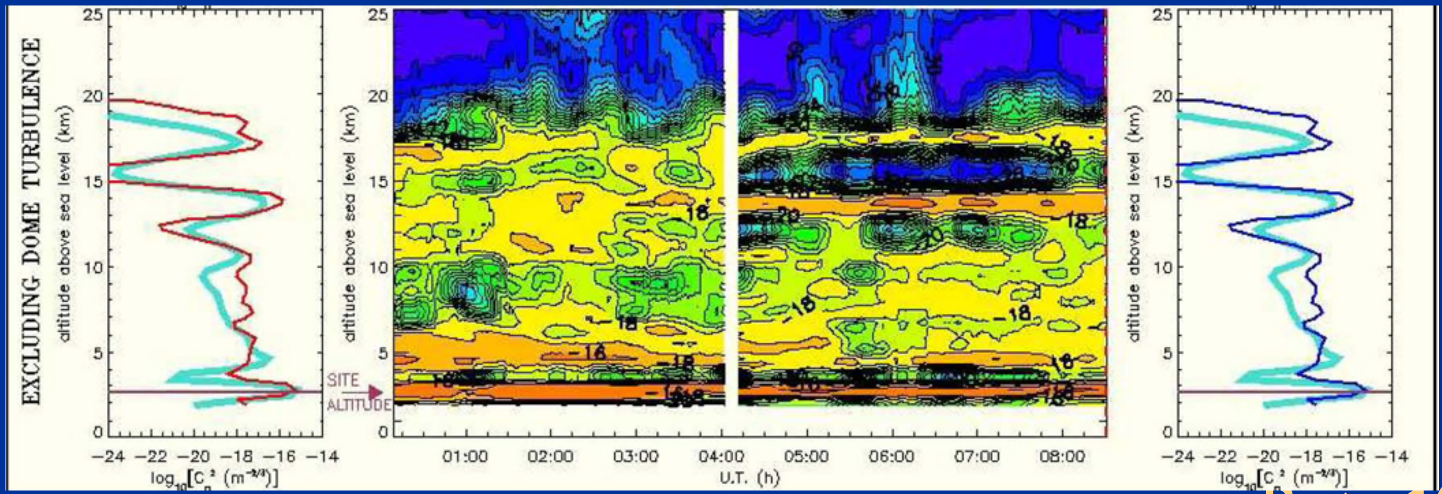
Turbulence arises in many places (part 1)



Two examples of measured atmospheric turbulence profiles

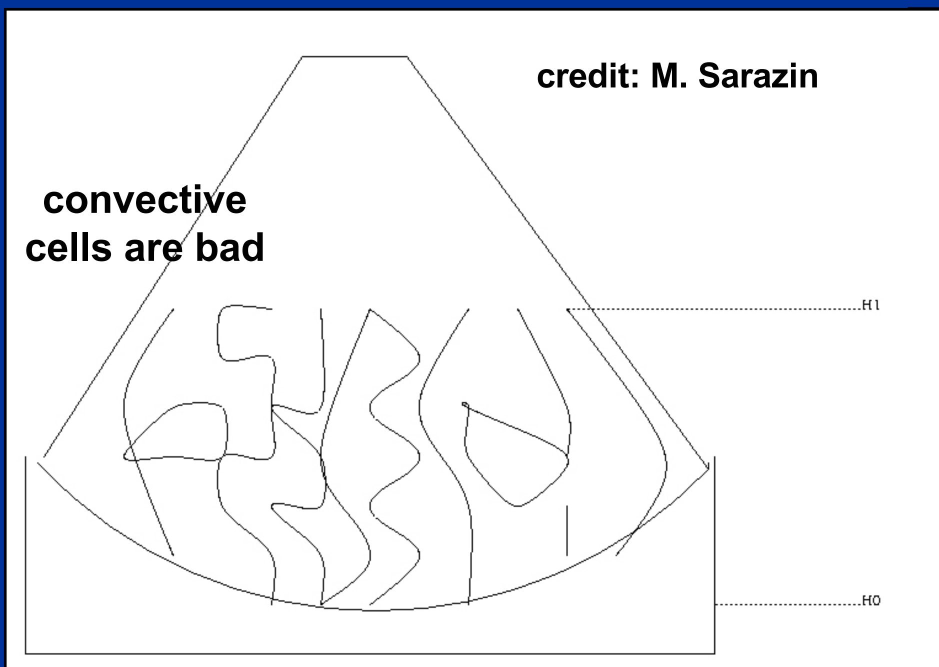


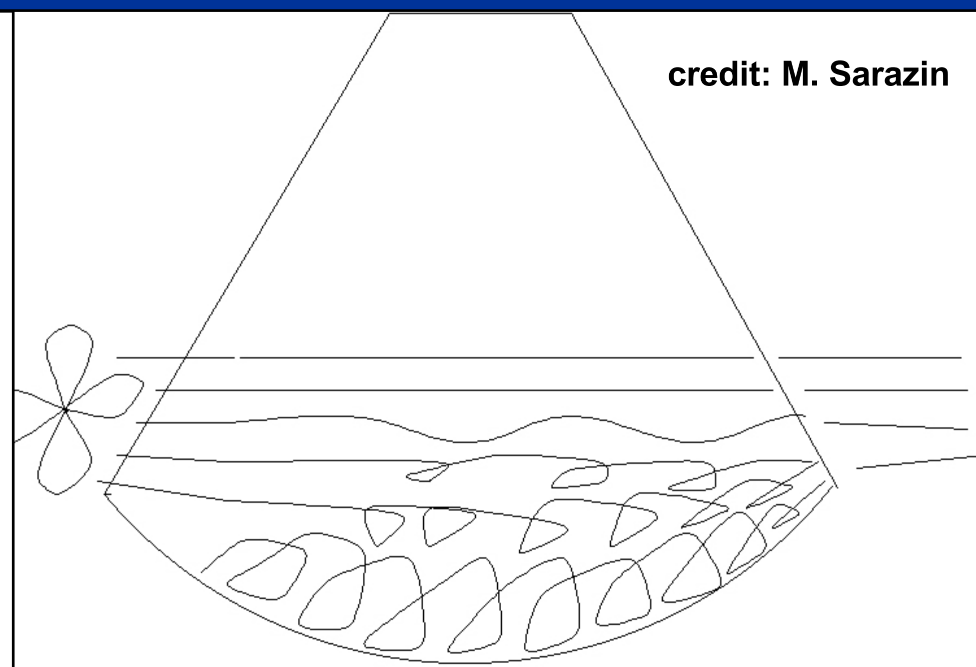
Credit: cute-SCIDAR group,
J. J. Fuensalida, PI



Turbulence within dome: "mirror seeing"

- When a mirror is warmer than dome air, convective equilibrium is reached.
- Remedies: Cool mirror itself, or blow air over it.





To control mirror temperature: dome air conditioning (day), blow air on back (night), send electric current through front Al surface-layer to equalize temperature between front and back of mirror

Natural flushing by wind can frequently reduce "mirror seeing"

- Natural flushing by wind.
  - Wind flushing decreases temperature differences, but increases dynamical effects.
  - Thus there is an optimal wind speed.

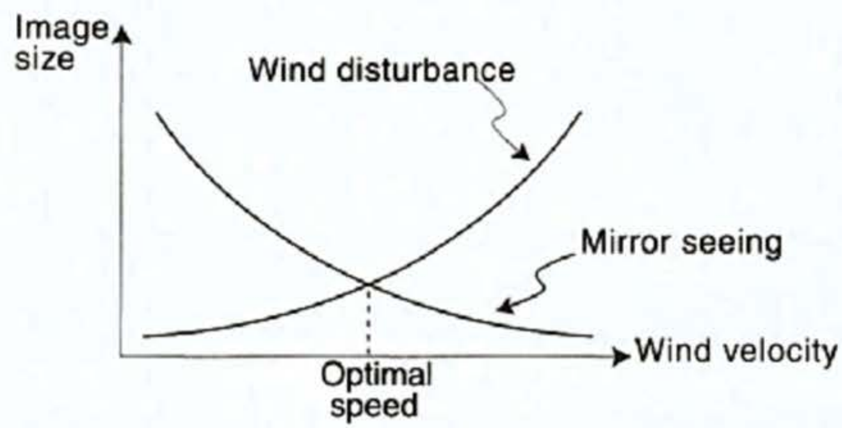


Fig. 9.10. Notional plot showing how mirror seeing improves with wind velocity while dynamic effects worsen. Optimal wind speed can be maintained by adjusting the enclosure's louvers and windscreen.

From Bely, *The Design and Construction of Large Optical Telescopes.*

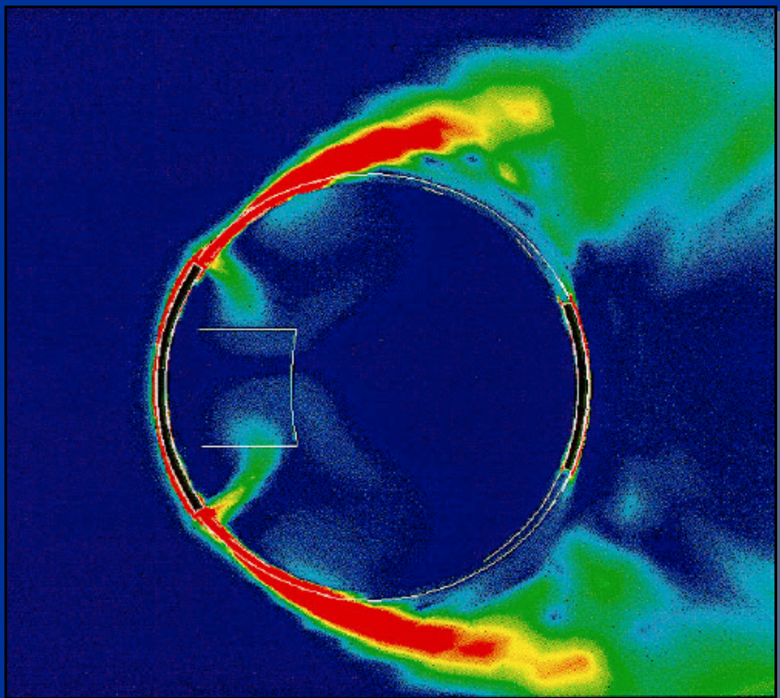
Credit: Majewski, Univ. of Virginia

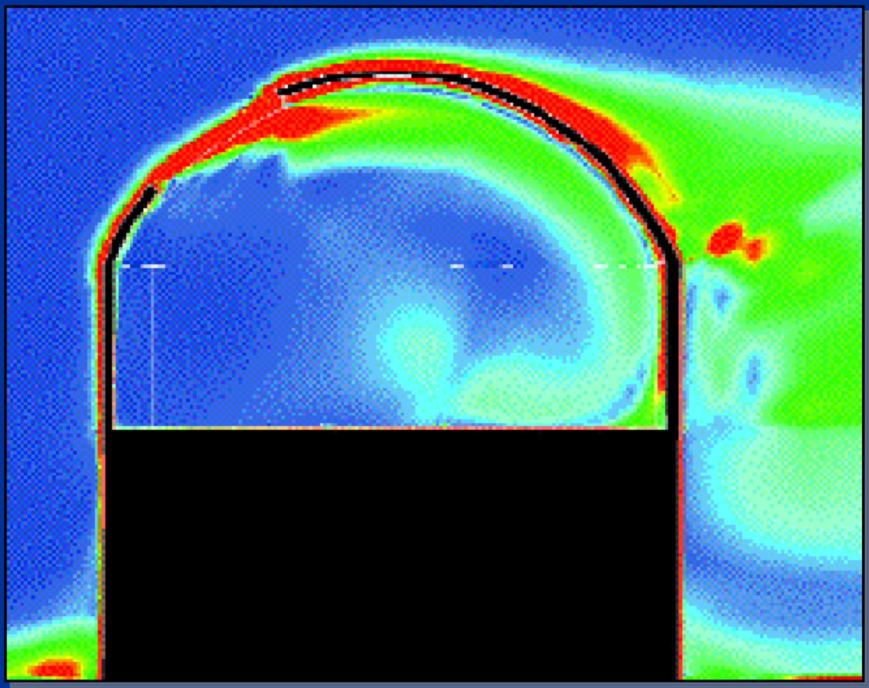
Modern observatory domes have louvres to let wind flow through

Gemini dome

VLT Domes

Turbulence also arises from wind flowing over the telescope dome





"Wake Turbulence" – you may not want to point telescope in direction opposite to wind, on windy night

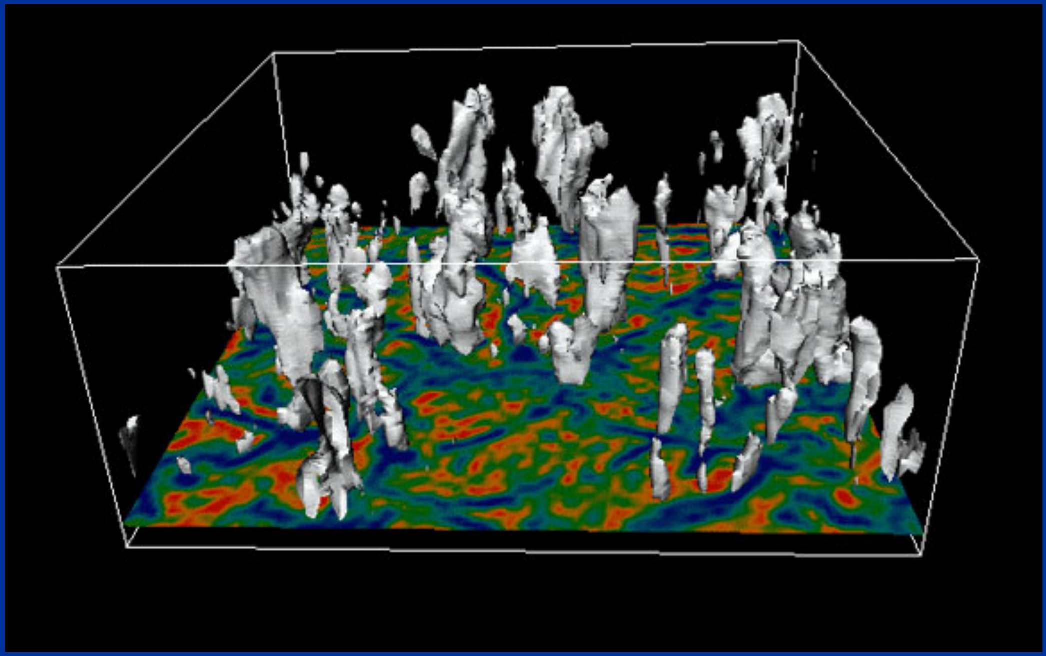
Computational fluid dynamics simulation (D. de Young)

Turbulent boundary layer has largest effect on "seeing"

- Wind speed must be zero at ground, must equal v_{wind} several hundred meters up (in the "free" atmosphere)
- Adjustment takes place at bottom of boundary layer
  - Where atmosphere feels strong influence of surface
  - Turbulent viscosity slows wind speed to zero
- Quite different between day and night
  - Daytime: boundary layer is thick (up to a km), dominated by convective plumes rising from hot ground. Quite turbulent.
  - Night-time: boundary layer collapses to a few hundred meters, is stably stratified. See a few "gravity waves." Perturbed if winds are high.

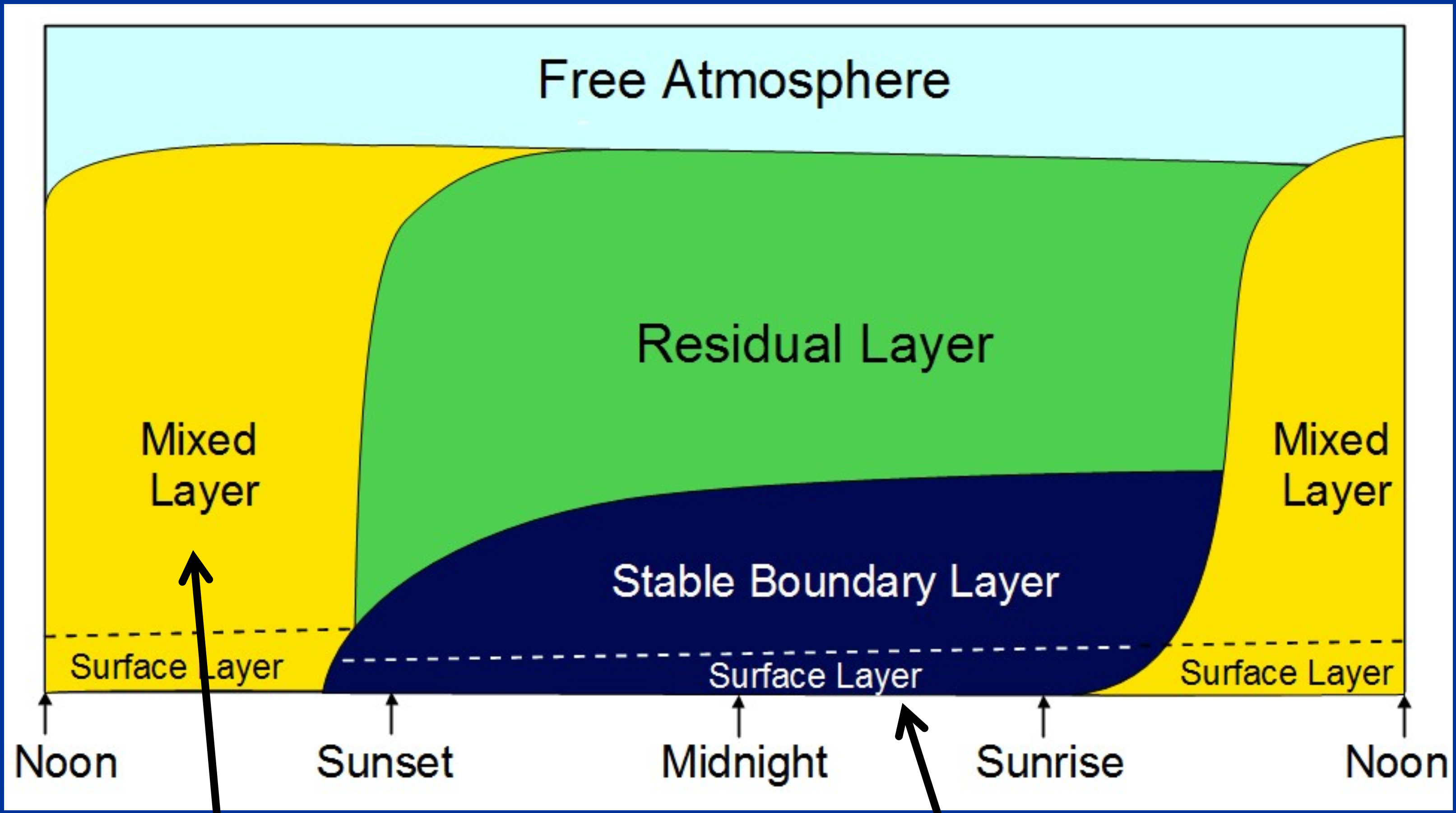
Convection takes place when temperature gradient is steep

- Daytime: ground is warmed by sun, air is cooler
- If temp. gradient between ground and ~ 1 km is steeper than "adiabatic gradient," warm volume of air raised upwards will have cooler surroundings, will keep rising
- These warm volumes of air carry thermal energy upwards



UCAR large eddy simulation of convective boundary layer

Boundary layer is much thinner at night: Day ~ 1 km, Night ~ few hundred meters



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Daytime convection

Surface layer: where viscosity is largest effect

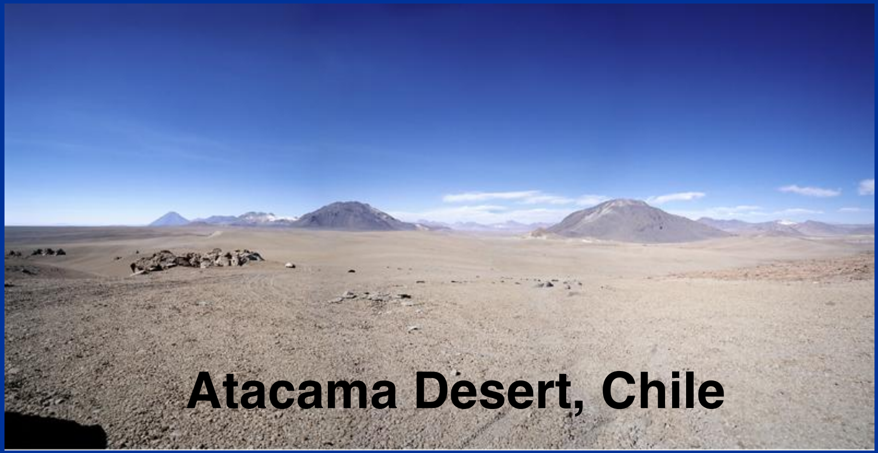
Implications: solar astronomers vs. night-time astronomers

- Daytime: Solar astronomers have to work with thick and messy turbulent boundary layer
- Night-time: Less total turbulence, but still the single largest contribution to “seeing”
- Neutral times: near dawn and dusk
  - Smallest temperature difference between ground and air, so wind shear causes smaller temperature fluctuations

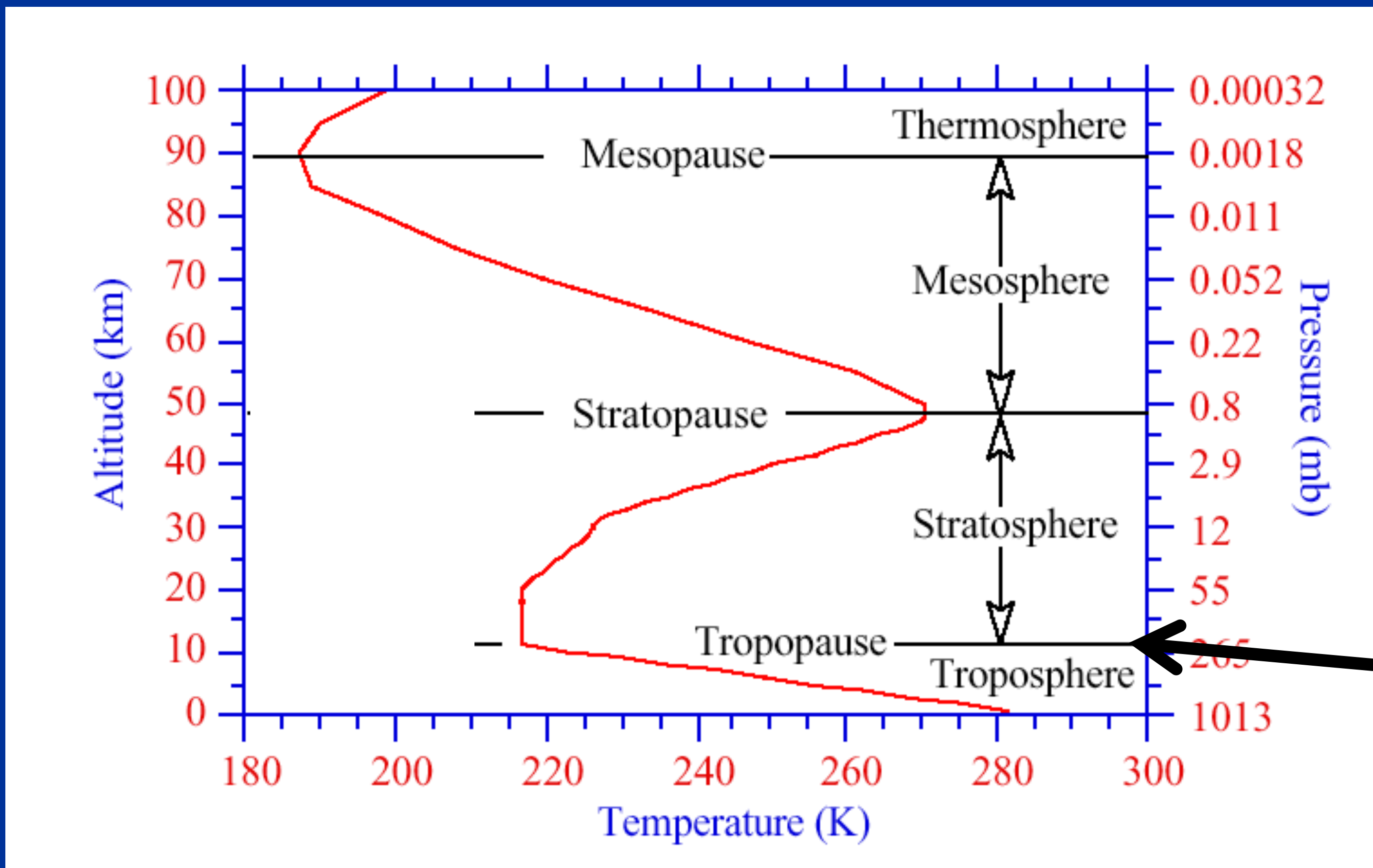
Concept Question

- Think of as many reasons as you can why high mountain tops have the best “seeing” (lowest turbulence). Prioritize your hypotheses from most likely to least likely.

- Use analogous reasoning to explain why the high Atacama Desert in Chile also has excellent “seeing”.



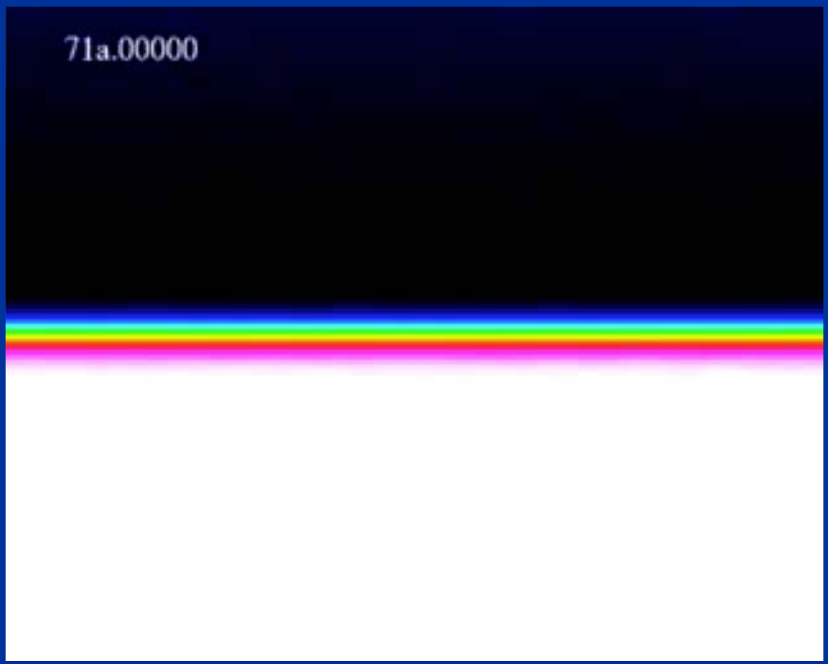
Turbulence in the "free atmosphere" above the boundary layer



Strong wind shear at tropopause

Wind shear mixes layers with different temperatures

- Wind shear → Kelvin Helmholtz instability



Computer simulation by Ceniceros and Roma, UCSB

- If two regions have different temperatures, temperature fluctuations δT will result
- T fluctuations $\Longrightarrow$ index of refraction fluctuations

Sometimes clouds show great Kelvin-Helmholtz vortex patterns

A clear sign of wind shear

Leonardo da Vinci's view of turbulence

Drawing of a turbulent flow by Leonardo da Vinci (1452–1519), who recognized that turbulence involves a multitude of eddies at various scales.

Kolmogorov turbulence in a nutshell

Big whorls have little whorls,

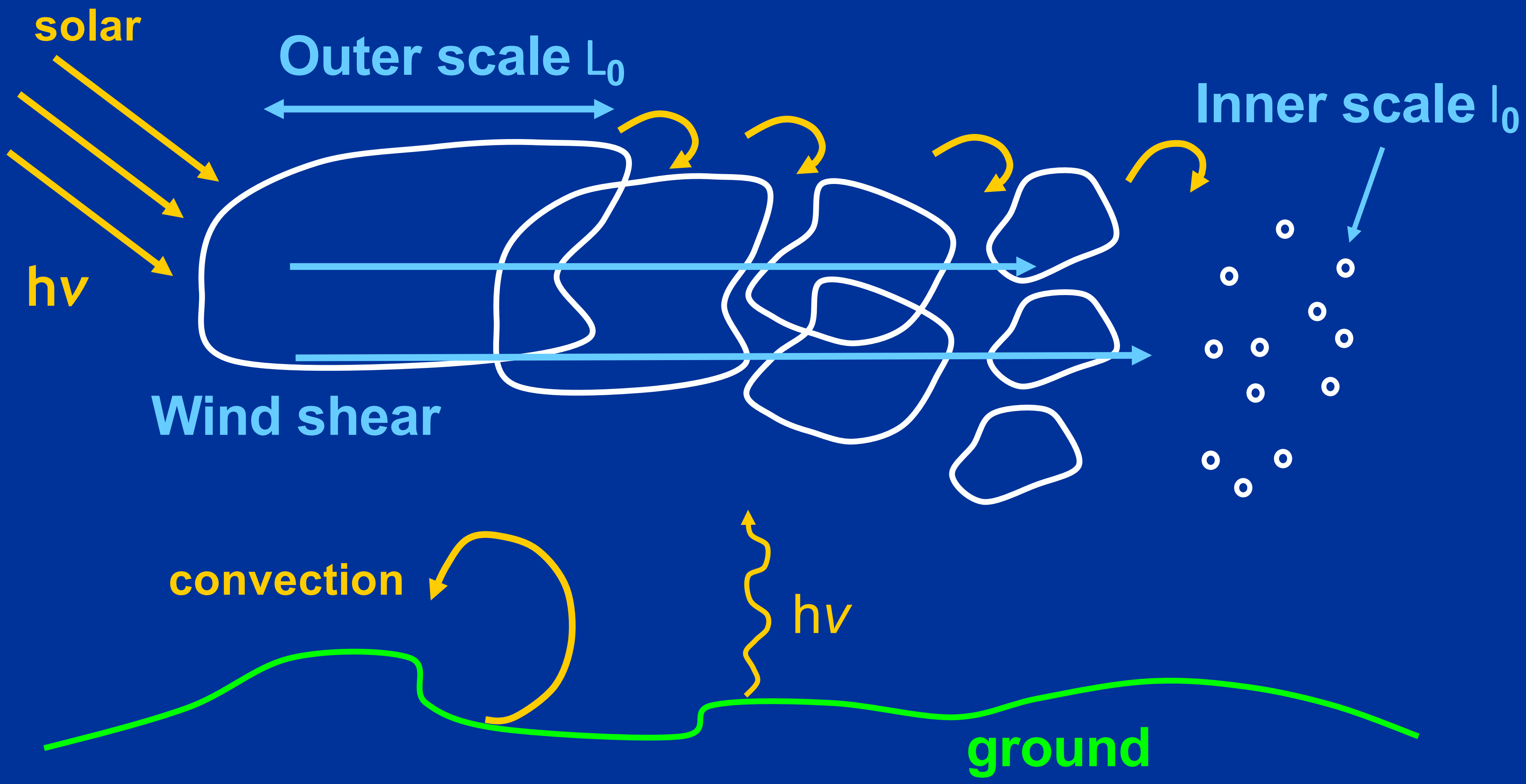
Which feed on their velocity;

Little whorls have smaller whorls,

And so on unto viscosity.

L. F. Richardson (1881-1953)

Kolmogorov turbulence, cartoon



Kolmogorov turbulence, in words

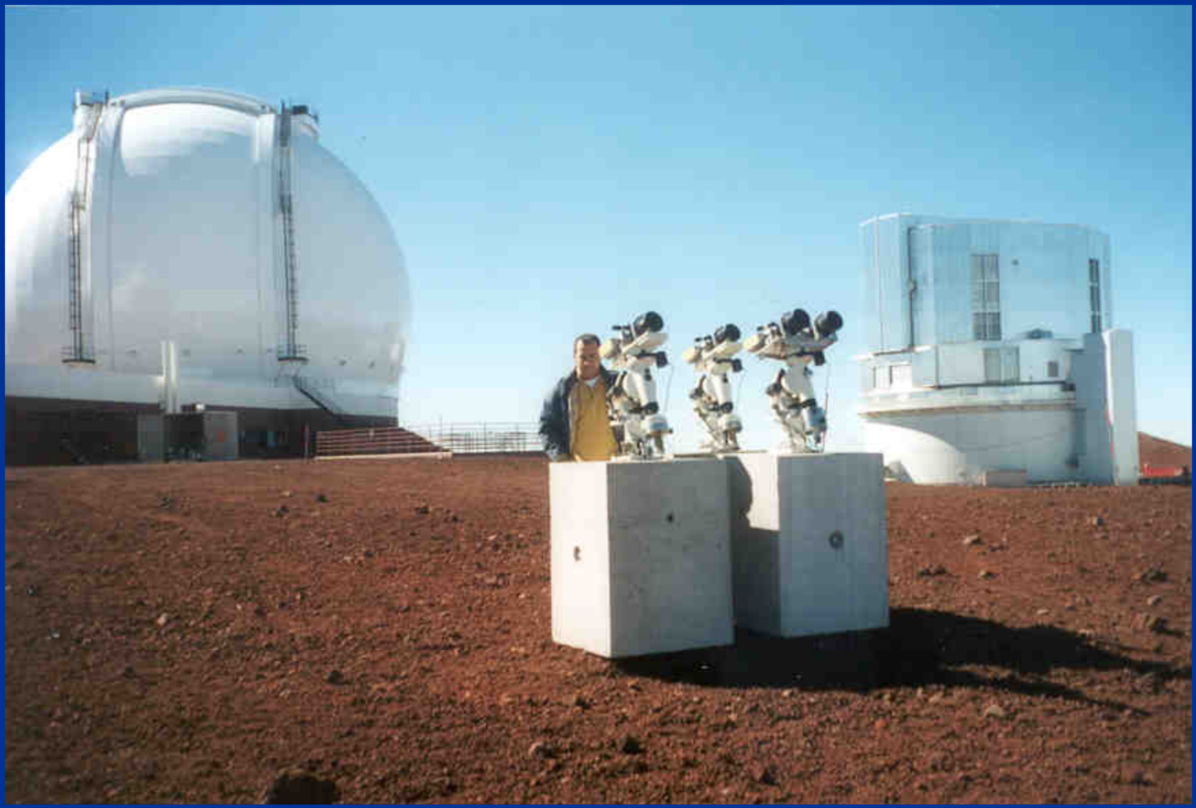
- Assume energy is added to system at largest scales - "outer scale" L_0
- Then energy cascades from larger to smaller scales (turbulent eddies "break down" into smaller and smaller structures).
- Size scales where this takes place: "Inertial range".
- Finally, eddy size becomes so small that it is subject to dissipation from viscosity. "Inner scale" l_0
- L_0 ranges from 10's to 100's of meters; l_0 is a few mm

Breakup of Kelvin-Helmholtz vortex

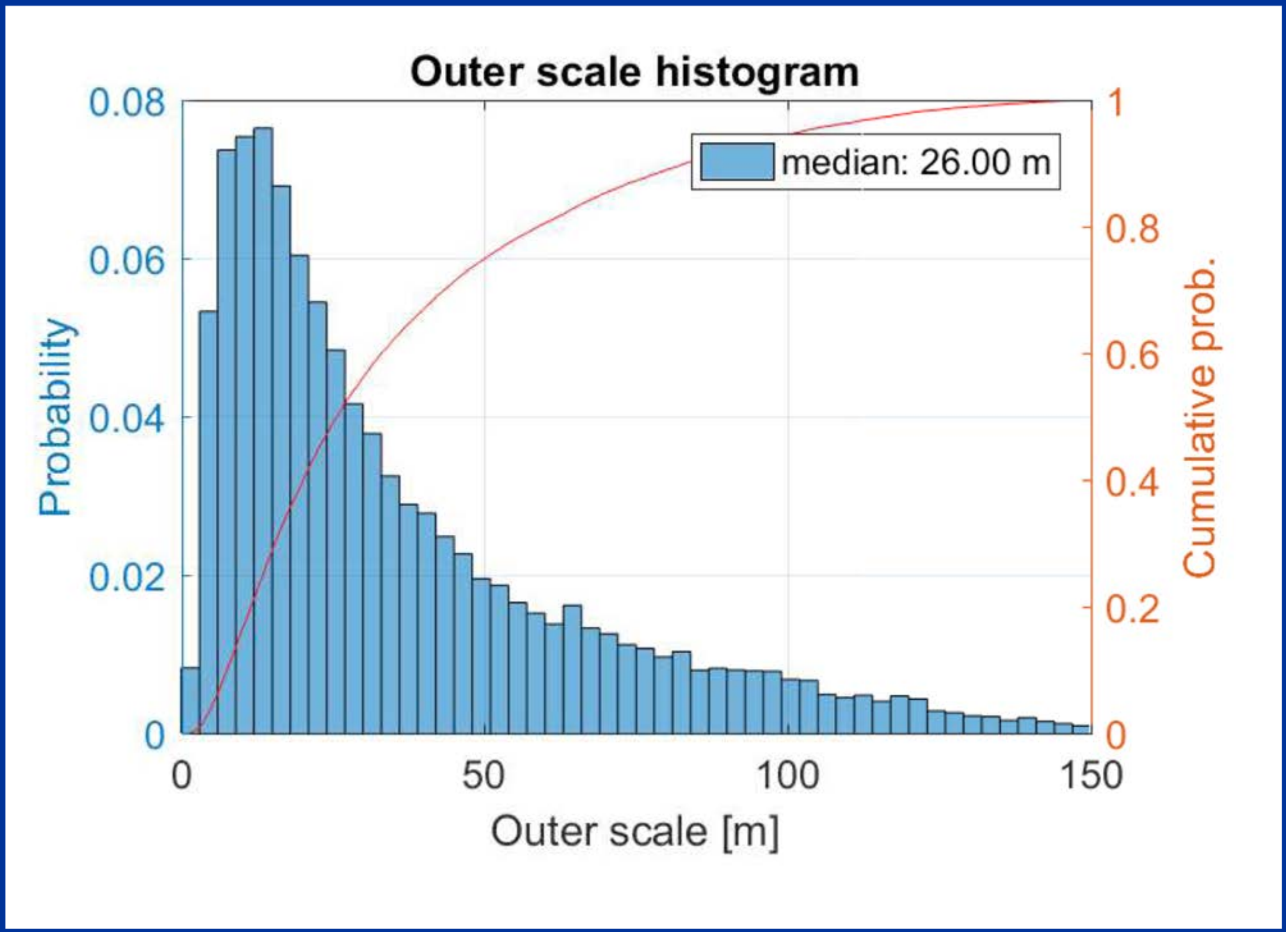
- Start with large coherent vortex structure, as is formed in K-H instability
- Watch it develop smaller and smaller substructure
- Analogous to Kolmogorov cascade from large eddies to small ones
- http://www.youtube.com/watch?v=hUXVHJoXMmU

How large is the Outer Scale?

- Dedicated instrument, the Generalized Seeing Monitor (GSM), built by Dept. of Astrophysics, Nice Univ.)



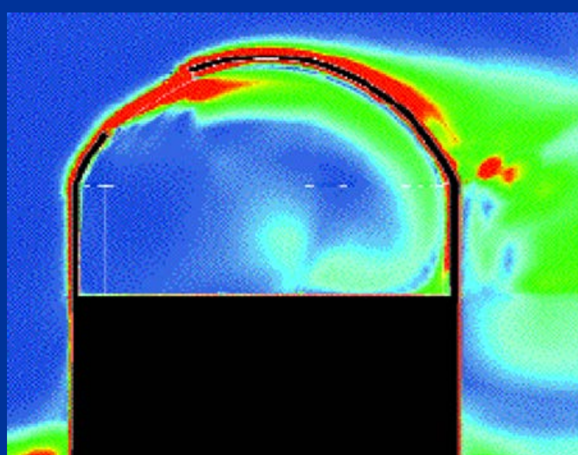
Median Outer Scale ~ 25-30 m, from Generalized Seeing Monitor measurements at Calern, Fr

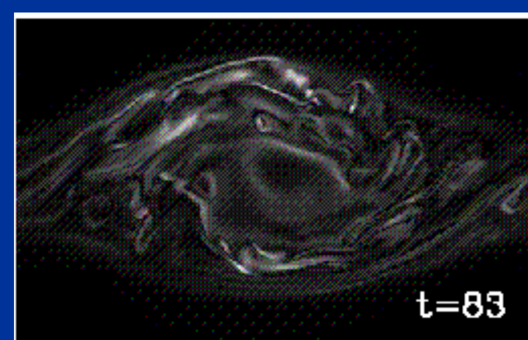


- Aristidi et al. 2019

Concept Question

- What do you think really determines the outer scale in the boundary layer? At the tropopause?
- Hints:





The Kolmogorov turbulence model, derived from dimensional analysis (1)

- v = velocity, ε = energy dissipation rate per unit mass, ν = viscosity, l_0 = inner scale, l = local spatial scale

- Energy/mass = $v^2/2 \sim v^2$

- Energy dissipation rate per unit mass

  $\varepsilon \sim v^2/\tau = v^2 / (l / v) = v^3 / l$

  $v \sim (\varepsilon\, l)^{1/3}$

  Energy $v^2 \sim \varepsilon^{2/3}\, l^{2/3}$

Kolmogorov Turbulence Model (2)

- **1-D power spectrum of velocity fluctuations:** $k = 2\pi / l$

$$\Phi(k)\,\Delta k \sim v^2 \sim (\varepsilon\, l)^{2/3} \sim \varepsilon^{2/3} k^{-2/3}$$ or, dividing by k,

$$\Phi(k) \sim k^{-5/3} \quad \text{(one dimension)}$$

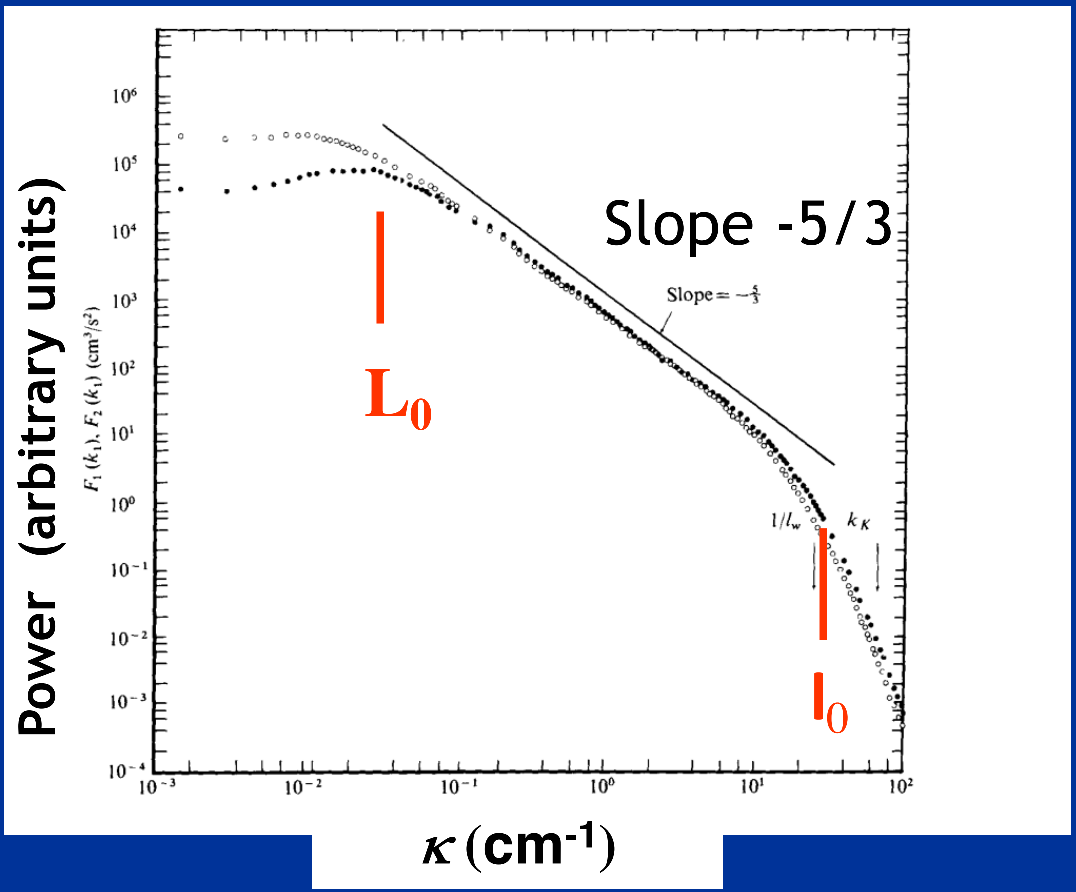
- **3-D power spectrum:** $\Phi^{3D}(k) \sim \Phi / k^2$

$$\Phi^{3D}(k) \sim k^{-11/3} \quad \text{(3 dimensions)}$$

- For a more rigorous calculation: V. I. Tatarski, 1961, "Wave Propagation in a Turbulent Medium", McGraw-Hill, NY

Lab experiments agree

- Air jet, 10 cm diameter (Champagne, 1978)
- Assumptions: turbulence is homogeneous, isotropic, stationary in time



The size of the inertial range is related to the Reynolds number

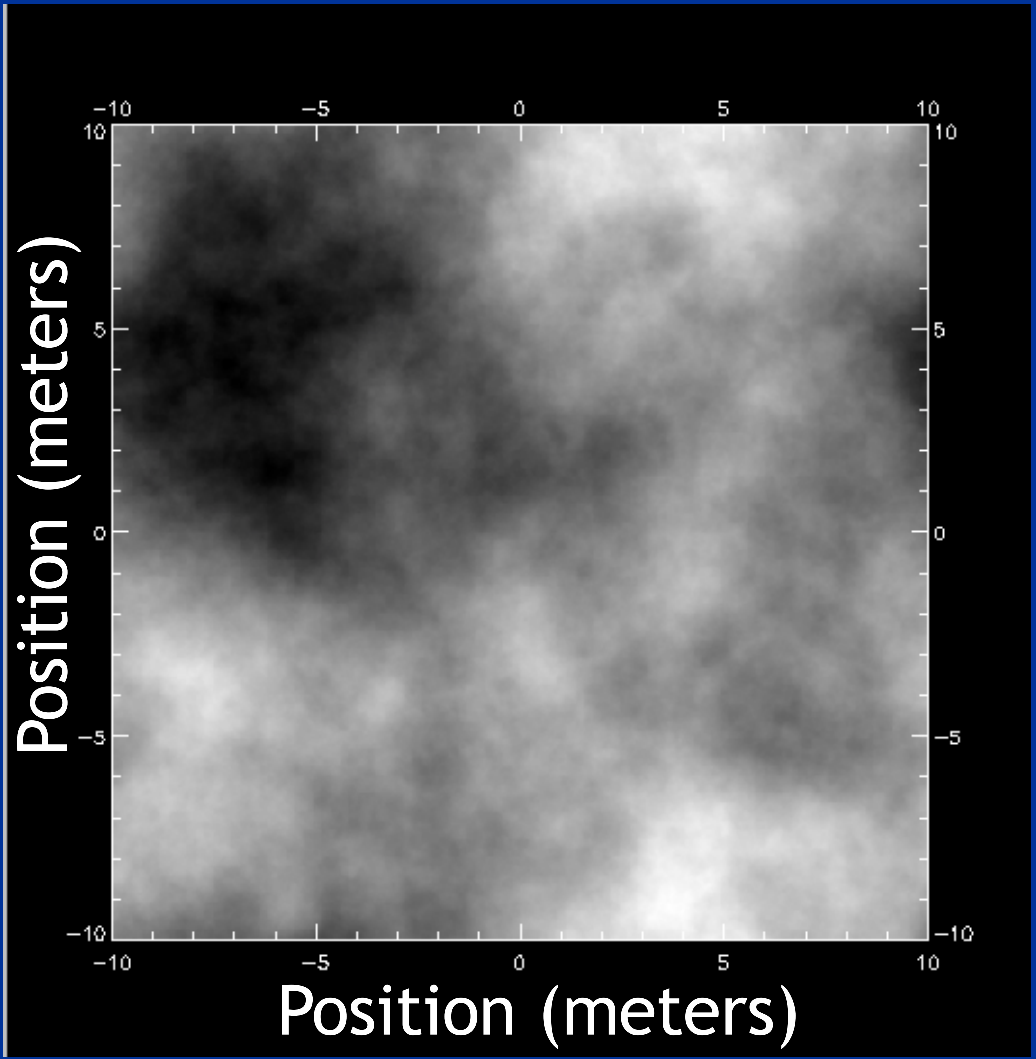
- Outer scale of turbulence: L_0
  - Size of the largest turbulent eddy
- Inner scale of turbulence: l_0
  - Below this scale, collisional viscosity wipes out any remaining velocity gradients
- Can show that

$$\frac{L_0}{l_0} \approx \left(\frac{\mathrm{v} L_0}{\nu} \right)^{3/4} \equiv (\mathrm{Re})^{3/4} \gg 1$$

where the Reynolds number $\mathrm{Re} \approx \dfrac{\text{inertial force}}{\text{viscous force}}$

- "Fully developed turbulence": Re > 5 x 10^3 (or more)

What does a Kolmogorov distribution of phase look like?



- A Kolmogorov "phase screen" courtesy of Don Gavel
- Shading (black to white) represents phase differences of ~1.5 μm
- r_0 = 0.4 meter

Structure functions are used a lot in AO discussions. What are they?

- Mean values of meteorological variables change over minutes to hours. Examples: T, p, humidity

- If $f(t)$ is a non-stationary random variable,

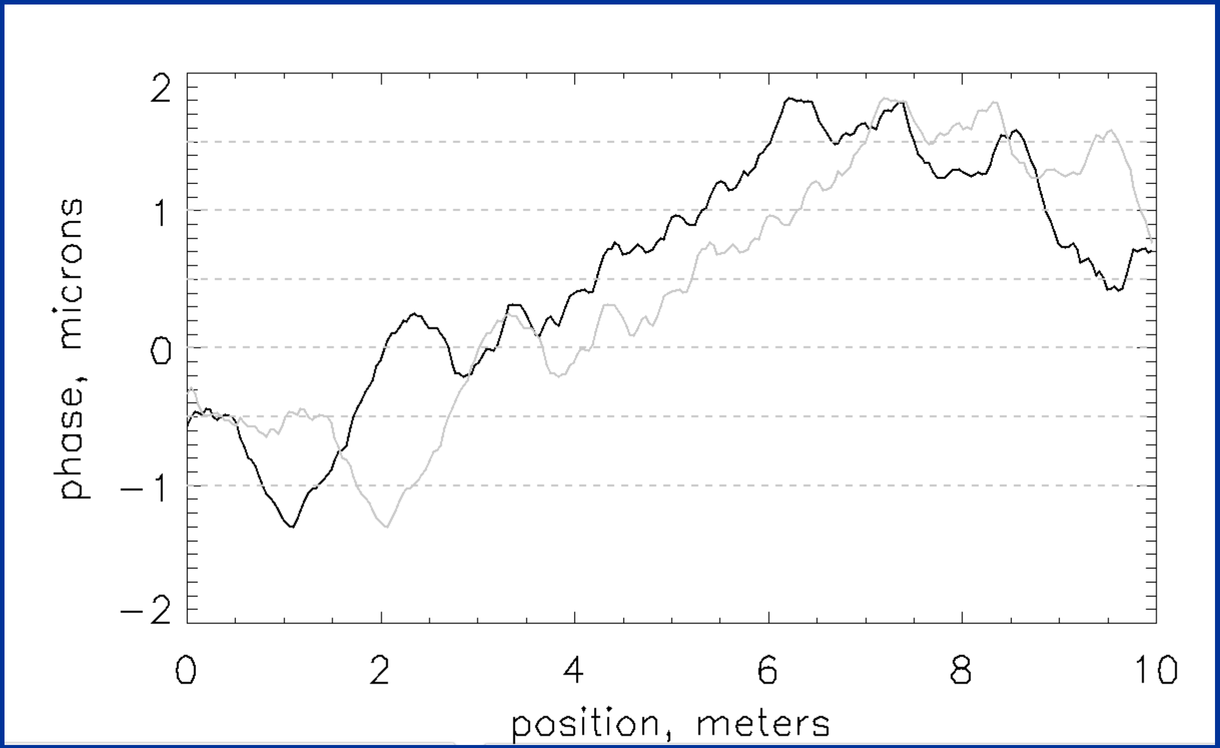
  $F_t(\tau) = f(t+\tau) - f(t)$ is a difference function that is stationary for small τ.

- Structure function is measure of intensity of fluctuations of $f(t)$ over a time scale less than or equal to τ :

$$D_f(\tau) = \langle [F_t(\tau)]^2 \rangle = \langle [f(t+\tau) - f(t)]^2 \rangle$$

More about the structure function (1)

$$D_\phi(\vec{r}) \equiv \left\langle \left| \phi(\vec{x}) - \phi(\vec{x}+\vec{r}) \right|^2 \right\rangle = \int_{-\infty}^{\infty} dx \ \left| \phi(\vec{x}) - \phi(\vec{x}+\vec{r}) \right|^2$$



Structure function for atmospheric fluctuations, Kolmogorov turbulence

- Scaling law we derived earlier: $v^2 \sim \varepsilon^{2/3}\, l^{2/3} \sim r^{2/3}$ where r is spatial separation between two points

- Heuristic derivation: Velocity structure function ~ v^2

$$D_v(r) \equiv \left\langle \left[v(x) - v(x+r) \right]^2 \right\rangle \propto r^{2/3} \quad \text{or} \quad D_v(r) = C_v^2 r^{2/3}$$

- Here C_v^2 = constant to clean up "look" of the equation

Derivation of D_v from dimensional analysis (1)

- If turbulence is homogenous, isotropic, stationary

$$D_v(x_1, x_2) = \alpha \times f(|\, x_1 - x_2 \,| / \beta)$$

where *f* is a dimensionless function of a dimensionless argument.

- Dimensions of α are v^2, dimensions of β are length, and they must depend only on ε and ν (the only free parameters in the problem).

$[\ \nu\ ] \sim cm^2\ s^{-1}$ $\qquad [\ \varepsilon\ ] \sim erg\ s^{-1}\ gm^{-1} \sim cm^2\ s^{-3}$

Derivation of D_v from dimensional analysis (2)

- The only combinations of ε and v with the right dimensions are

$$\alpha = v^{1/2}\varepsilon^{1/2}$$

$$\text{dimensions } cm\ s^{-1/2} \times cm\ s^{-3/2} = (cm/s)^2$$

$$\text{and } \beta = v^{3/4}\varepsilon^{-1/4}$$

$$\text{dimensions } (cm^{3/2}\ s^{-3/4}) \times (s^{3/4} cm^{-1/2}) = cm$$

$$D_v = v^{1/2}\varepsilon^{1/2} f(|x_1 - x_2| / v^{3/4}\varepsilon^{-1/4})$$

For f to be dimensionless, must have $f(x) = x^{2/3}$

$$\Rightarrow D_v = \varepsilon^{2/3} |x_1 - x_2|^{2/3} \equiv C_v^2 |x_1 - x_2|^{2/3}$$

What about temperature and index of refraction fluctuations?

- Temperature fluctuations are carried around passively by velocity field (incompressible fluids).
- So T and N have structure functions similar to v:

$$D_T(r) = \langle [T(x) - T(x+r)]^2 \rangle = C_T^2 \, r^{2/3}$$

$$D_N(r) = \langle [N(x) - N(x+r)]^2 \rangle = C_N^2 \, r^{2/3}$$

How do you measure index of refraction fluctuations in situ?

- Refractivity $$N = (n-1)\times 10^{6} = 77.6\times(P/T)$$

- Index fluctuations $$\delta N = -77.6\times(P/T^{2})\delta T$$

$$C_N = \left(\partial N/\partial T\right)C_T = -77.6\times\left(P/T^{2}\right)C_T$$

$$C_N^2 = \left(77.6P/T^{2}\right)^{2} C_T^2$$

- So measure δT , p, and T; calculate C_N^2

Simplest way to measure C_N^2 is to use fast-response thermometers

$$D_T (r) = < [T (x) - v (T + r)]^2 > = C_T^2 \ r^{2/3}$$

- Example: mount fast-response temperature probes at different locations along a bar:

- Form spatial correlations of each time-series *T(t)*

Assumptions of Kolmogorov turbulence theory

- Medium is incompressible
- External energy is input on largest scales (only), dissipated on smallest scales (only)
  - Smooth cascade
- Valid only in inertial range $l \ll L_0$
- Turbulence is
  - Homogeneous
  - Isotropic

  } **Questionable**
- In practice, Kolmogorov model works surprisingly well!

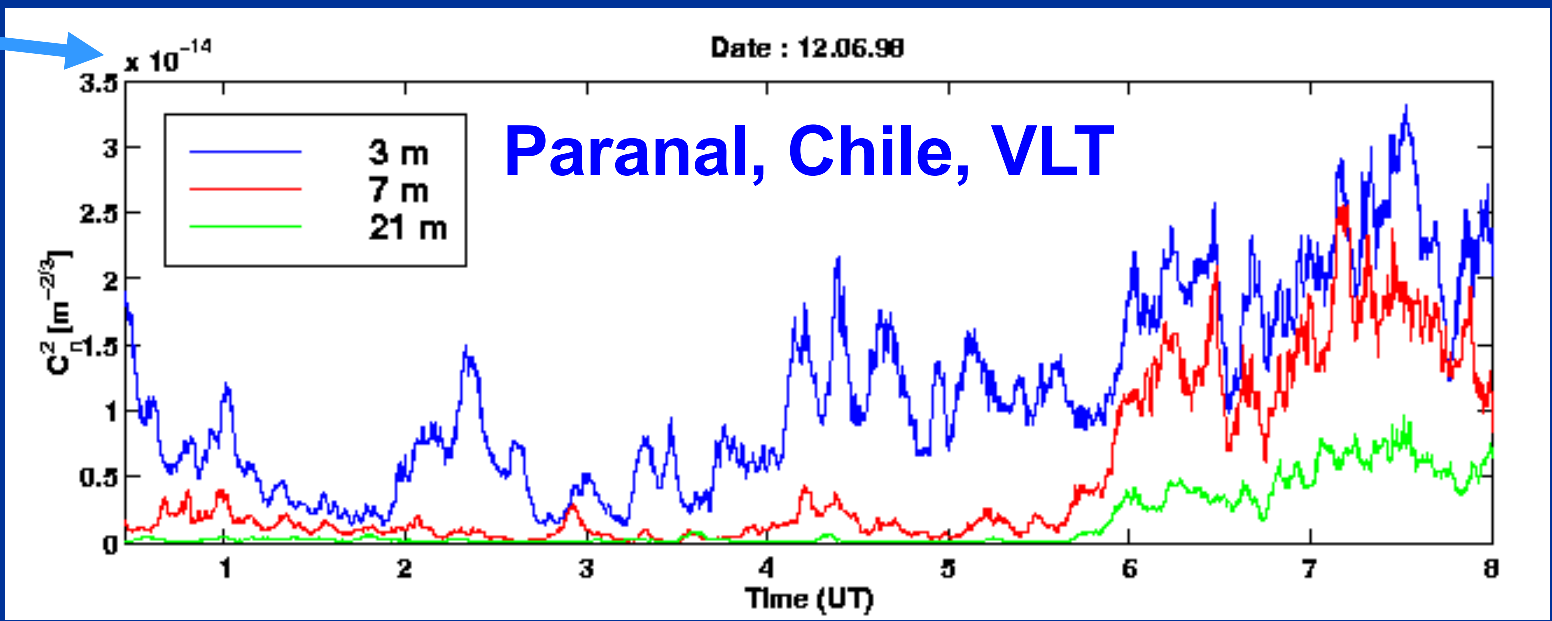
Typical values of C_N^2

- Index of refraction structure function

$$D_N(r) = \langle [N(x) - N(x+r)]^2 \rangle = C_N^2 \, r^{2/3}$$

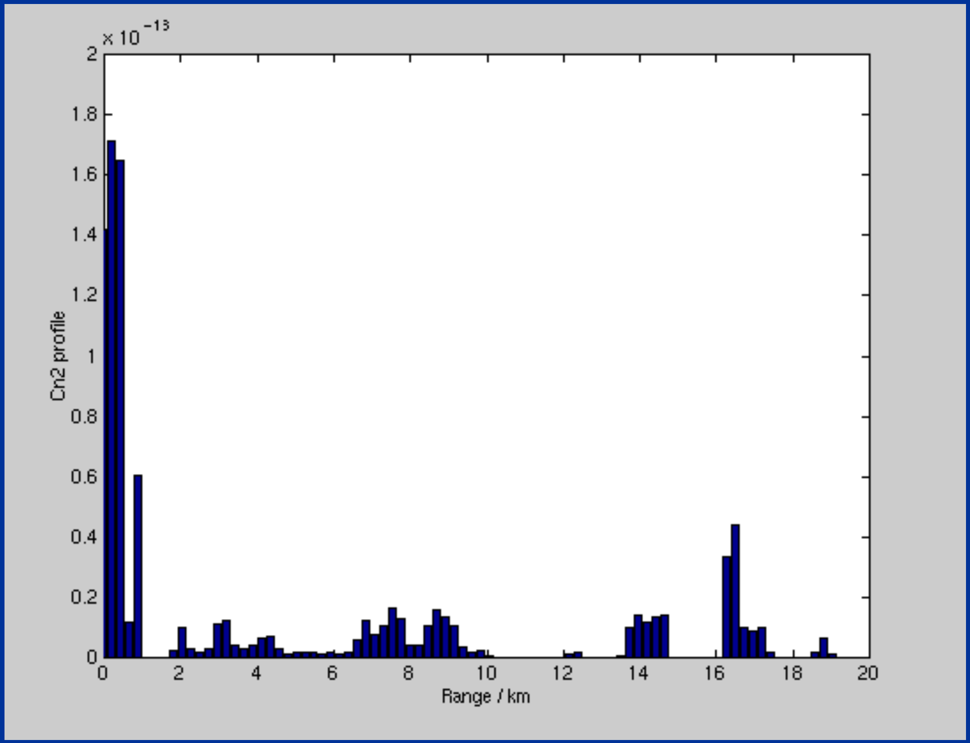
- Night-time boundary layer: $C_N^2 \sim 10^{-13} - 10^{-15} \; m^{-2/3}$

10^{-14}

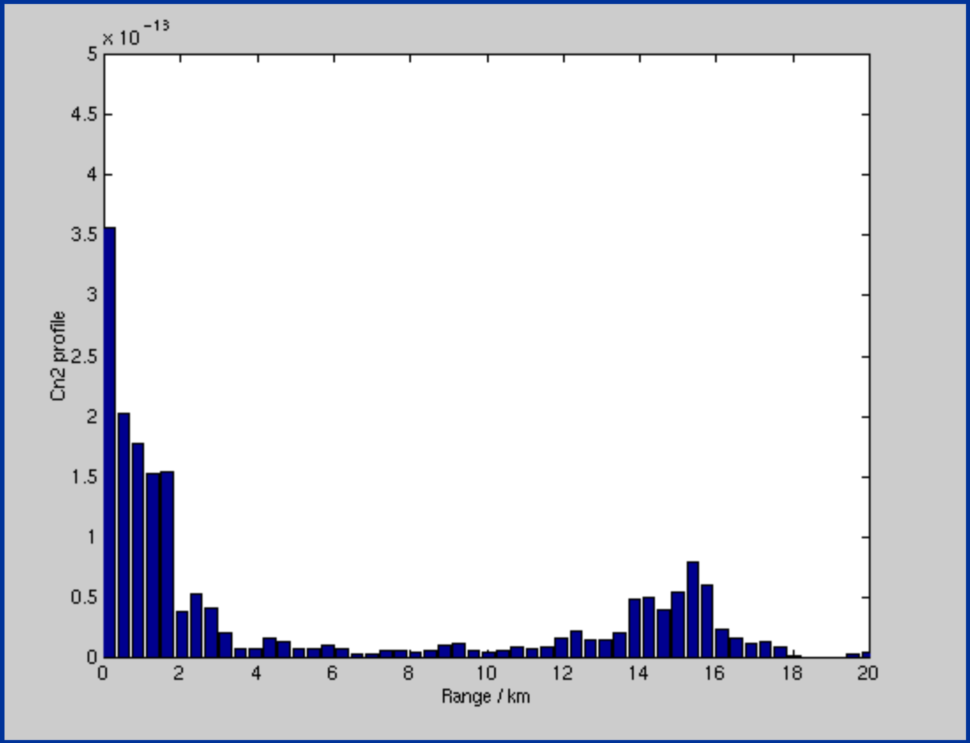


Turbulence profiles from SCIDAR

Eight minute time period (C. Dainty, NUI)



Siding Spring, Australia



Starfire Optical Range, Albuquerque NM

Atmospheric Turbulence: Main Points

- Dominant locations for index of refraction fluctuations: atmospheric boundary layer and tropopause
- Atmospheric turbulence (mostly) obeys Kolmogorov statistics
- Kolmogorov turbulence is derived from dimensional analysis (heat flux in = heat flux in turbulence)
- Structure functions derived from Kolmogorov turbulence:

$$D_N(r) \equiv \left\langle \left[N(x) - N(x+r) \right]^2 \right\rangle \propto r^{2/3} \quad \text{or} \quad D_N(r) = C_N^2 r^{2/3}$$

- All else will follow from these points!

Part 2: Effect of turbulence on spatial coherence function of light

- We will use structure functions $D \sim r^{2/3}$ to calculate various statistical properties of light propagation thru index of refraction variations

Definitions - Structure Function and Correlation Function

- ***Structure function: Mean square difference***

$$D_{\phi}(\vec{r}) \equiv \left\langle \left| \phi(\vec{x}) - \phi(\vec{x}+\vec{r}) \right|^2 \right\rangle = \int_{-\infty}^{\infty} dx \ \left| \phi(\vec{x}) - \phi(\vec{x}+\vec{r}) \right|^2$$

- ***Covariance function: Spatial correlation of a function with itself***

$$B_{\phi}(\vec{r}) \equiv \left\langle \phi(\vec{x}+\vec{r})\phi(\vec{x}) \right\rangle = \int_{-\infty}^{\infty} dx \ \phi(\vec{x}+\vec{r})\phi(\vec{x})$$

Relation between structure function and covariance function

$$D_{\phi}(\vec{r}) = 2\left[B_{\phi}(0) - B_{\phi}(\vec{r}) \right]$$

Structure function

Covariance function

- A problem on future homework:
  - Derive this relationship
  - Hint: expand the product in the definition of $D_{\phi}(r)$ and assume homogeneity to take the averages

Definitions - Spatial Coherence Function

For light wave $\Psi = \exp[i\phi(\vec{x})]$, phase is $\phi(\vec{x}) = kz - \omega t$

- Spatial coherence function of field is defined as

$$C_\Psi(\vec{r}) \equiv \left\langle \Psi(\vec{x})\Psi^*(\vec{x}+\vec{r}) \right\rangle$$

Covariance for complex fn's

  - Note that Hardy calls this function $B_h(\vec{r})$ but I've called it $C_\psi(r)$ in order to avoid confusion with the correlation function $B_\phi(r)$. $C_\psi(r)$ is a measure of how "related" the field Ψ is at one position (e.g. x) to its values at neighboring positions (say $x + r$).

- Since $\Psi(\vec{x}) = \exp[i\phi(\vec{x})]$ and $\Psi^*(\vec{x}) = \exp[-i\phi(\vec{x})]$,

$$C_\Psi(\vec{r}) = \left\langle \exp i[\phi(\vec{x}) - \phi(\vec{x}+\vec{r})] \right\rangle$$

Now evaluate spatial coherence function C_ψ (r)

- For a Gaussian random variable χ with zero mean, it can be shown that

$$\langle \exp i\chi \rangle = \exp\left(- \langle \chi^2 \rangle \;/\; 2\right)$$

- So

$$C_\Psi(\vec{r}) = \langle \exp i[\phi(\vec{x}) - \phi(\vec{x}+\vec{r})] \rangle$$
$$= \exp\left[-\left\langle \left|\phi(\vec{x}) - \phi(\vec{x}+\vec{r})\right|^2 \right\rangle / 2\right] \equiv \exp\left[-D_\phi(\vec{r})/2\right]$$

- So finding spatial coherence function C_ψ *(r)* amounts to evaluating the structure function for phase D_ϕ *(r) !*

Solve for $D_\varphi(r)$ in terms of the turbulence strength C_N^2 (1)

- We want to evaluate $B_h(\vec{r}) = \exp\left[-D_\phi(\vec{r})/2\right]$

- Recall that $D_\phi(\vec{r}) = 2\left[B_\phi(0) - B_\phi(\vec{r})\right]$

- So next we need to know the phase covariance:

$$B_\phi(\vec{r}) \equiv \left\langle \phi(\vec{x})\ \phi(\vec{x}+\vec{r})\right\rangle$$

Solve for $D_\varphi(r)$ in terms of the turbulence strength C_N^2 (2)

- But $\phi(\vec{x}) = k \int_h^{h+\delta h} dz \times n(\vec{x}, z)$ for a wave propagating vertically (in z direction) from height *h* to height *h* + *δh*.

- Here n(x, z) is the index of refraction.

- Hence $B_\phi(\vec{r}) = k^2 \int_h^{h+\delta h} dz' \int_h^{h+\delta h} dz'' \langle n(\vec{x}, z') n(\vec{x} + \vec{r}, z'') \rangle$

Solve for $D_\varphi(r)$ in terms of the turbulence strength C_N^2 (3)

- Change variables: $z = z'' - z'$

- Then $$B_\phi(\vec{r}) = k^2 \int_h^{h+\delta h} dz' \int_{h-z'}^{h+\delta h - z'} dz \, \langle n(\vec{x}, z') n(\vec{x}+\vec{r}, z'+z) \rangle$$
$$= k^2 \int_h^{h+\delta h} dz' \int_{h-z'}^{h+\delta h - z'} dz \, B_N(\vec{r}, z)$$

$$B_\phi(\vec{r}) = k^2 \delta h \int_{h-z'}^{h+\delta h - z'} dz B_N(\vec{r}, z) \cong k^2 \delta h \int_{-\infty}^{\infty} dz B_N(\vec{r}, z)$$

Solve for $D_\varphi(r)$ in terms of the turbulence strength C_N^2 (4)

- Now we can evaluate phase structure function $D_\varphi(r)$

$$D_\phi(\vec{r}) = 2\left[B_\phi(0) - B_\phi(\vec{r})\right] = 2k^2\delta h\int_{-\infty}^{\infty} dz\left[B_N(0,z) - B_N(\vec{r},z)\right]$$

$$D_\phi(\vec{r}) = 2k^2\delta h\int_{-\infty}^{\infty} dz\left\{\left[B_N(0,0) - B_N(\vec{r},z)\right] - \left[B_N(0,0) - B_N(0,z)\right]\right\}$$

$$D_\phi(\vec{r}) = k^2\delta h\int_{-\infty}^{\infty} dz\left[D_N(\vec{r},z) - D_N(0,z)\right]$$

Solve for $D_\varphi(r)$ in terms of the turbulence strength C_N^2 (5)

- But $D_N(\vec{r}) = C_N^2 \, |\, \vec{r}\, |^{2/3} = C_N^2 \left(r^2 + z^2\right)^{1/3}$ so

$$D_\phi(\vec{r}) = k^2 \delta h C_N^2 \int_{-\infty}^{\infty} dz \left[\left(r^2 + z^2\right)^{1/3} - z^{2/3}\right]$$

$$\left(\frac{2}{5}\frac{\Gamma(1/2)\Gamma(1/6)}{\Gamma(2/3)}\right) r^{5/3} = 2.914\, r^{5/3}$$

$$D_\phi(\vec{r}) = 2.914\, k^2 r^{5/3} C_N^2 \;\delta h \rightarrow 2.914\, k^2 r^{5/3} \int_0^{\infty} dh\, C_N^2(h)$$

Finally we can evaluate the spatial coherence function C_ψ (r) (!)

$$C_\Psi(\vec{r}) = \exp\left[-D_\phi(\vec{r})/2\right] = \exp\left[-\frac{1}{2}\left(2.914\, k^2 r^{5/3} \int_0^\infty dh\; C_N^2(h)\right)\right]$$

For a slant path you can add factor *(sec θ)*$^{5/3}$ to account for dependence on zenith angle *θ*

Concept Question: Note the scaling of the coherence function with separation, wavelength, turbulence strength. Think of a physical reason for each.

Given the spatial coherence function, calculate effect on telescope resolution

Outline of derivation:

- Define optical transfer functions of telescope, atmosphere
- Define r_0 as the telescope diameter where the two optical transfer functions are equal
  - $OTF_{telescope} = OTF_{atmosphere}$
- Calculate expression for r_0

Define optical transfer function (OTF)

- Imaging in the presence of imperfect optics (or aberrations in atmosphere): in intensity units

Image = Object $\otimes$ Point Spread Function

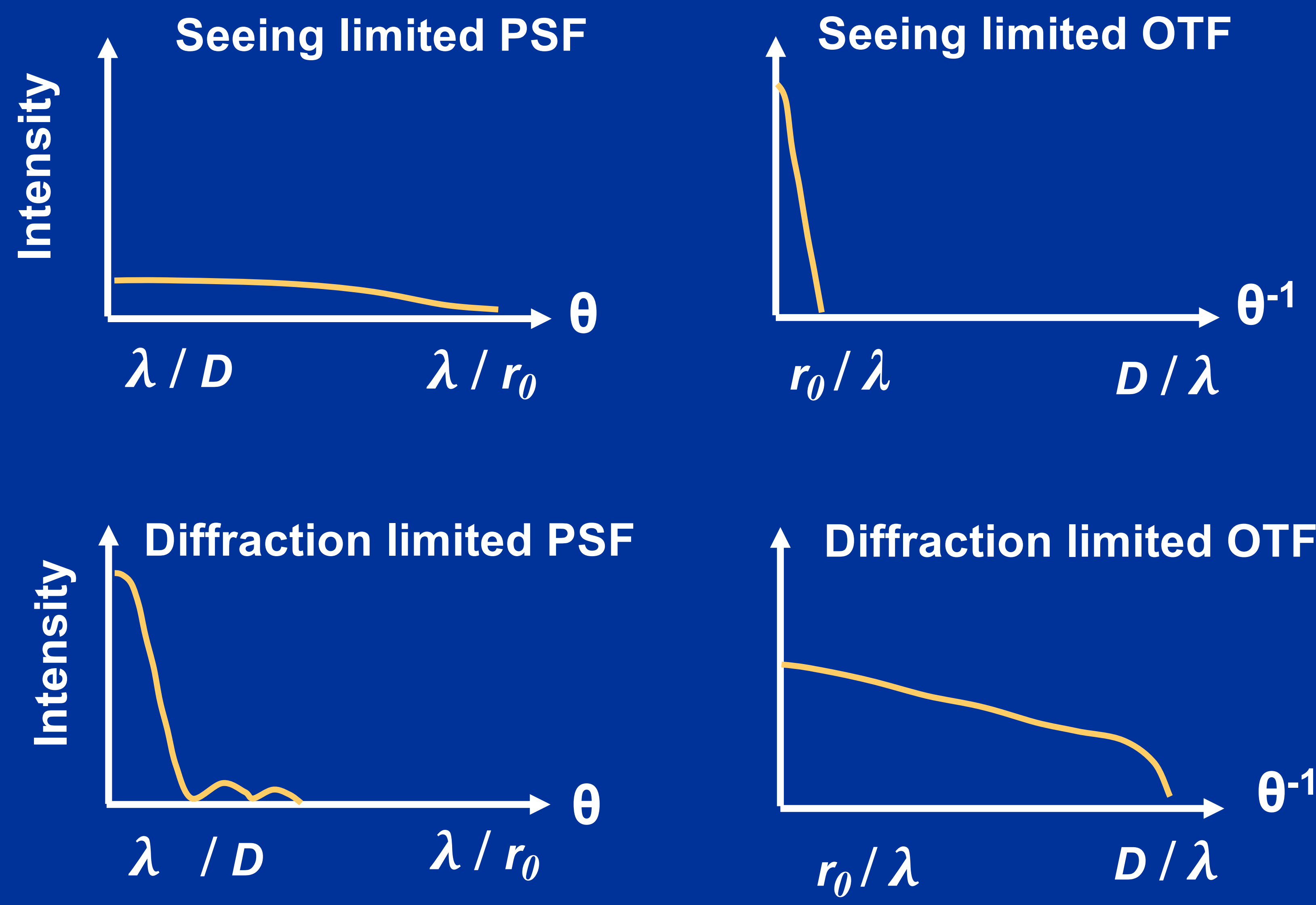
↖ convolved with

$$I = O \otimes PSF \equiv \int d\vec{x}\ O(\vec{x}-\vec{r})\ PSF\ (\vec{x})$$

- Take Fourier Transform: $\tilde{F}(I) = \tilde{F}(O)\ \tilde{F}(PSF)$
- Optical Transfer Function = Fourier Transform of PSF

$$\tilde{F}(I) = \tilde{F}(O) \times \text{OTF}$$

Examples of PSF's and their Optical Transfer Functions



Next time: Derive r_0 and all the good things that come from knowing r_0

- Define r_0 as the telescope diameter where the optical transfer functions of the telescope and atmosphere are equal
- Use r_0 to derive relevant timescales of turbulence
- Use r_0 to derive "Isoplanatic Angle":
  - AO performance degrades as astronomical targets get farther from guide star